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Spherically symmetric solutions in Covariant Horava-Lifshitz Gravity

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Abstract

We study the most general case of spherically symmetric vacuum solutions in the framework of the Covariant Horava Lifshitz Gravity, for an action that includes all possible higher order terms in curvature which are compatible with power-counting normalizability requirement. We find that solutions can be separated into two main classes: *(i)* solutions with nonzero radial shift function, and *(ii)* solutions with zero radial shift function. In the case *(ii)*, spherically symmetric solutions are consistent with observations if we adopt the view of Horava and Melby-Tomson [41], according to which the auxiliary field A can be considered as a part of an effective general relativistic metric, which is valid only in the IR limit. On the other hand, in the case *(i)*, consistency with observations implies that the field A should be independent of the spacetime geometry, as the Newtonian potential arises from the nonzero radial shift function. Also, our aim in this paper is to discuss and compare these two alternative but different assumptions for the auxiliary field A .

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1 Introduction

A recent higher order space derivative model for Gravity was formulated by Horava [1]. This model is power-counting renormalizable and serves as an ultraviolet (UV) completion of General Relativity (GR). This scenario is based on an anisotropy between space and time coordinates, which is expressed via the scalings $t \rightarrow b^z t$ and $x \rightarrow bx$, where z is a dynamical critical exponent. It is worth noting that Horava-Lifshitz (HL) Gravity has stimulated an extended research on Cosmology and black hole solutions, see for example [2]-[27], and we note that quantum field theory models in flat anisotropic space-time were also developed, see for example [28]-[33] and references therein.

In HL Gravity, the four-dimensional diffeomorphism invariance of GR is sacrificed in order to achieve power-counting renormalizability. Although HL Gravity violates local Lorentz invariance in the UV, GR is expected to be recovered in the infrared (IR) limit. This implies a very special renormalization group flow for the couplings of the model, in particular it is expected that the coupling λ in the extrinsic curvature term of the action has the behavior $\lambda \rightarrow 1$, i.e. that it flows towards its GR value. But there is no theoretical study supporting this specific behavior. In addition, there are several other potential inconsistencies in HL Gravity which have been discussed (see for example [34]-[40] and references therein). More specifically, the breaking of 4D diffeomorphism invariance introduces an additional scalar mode which may lead to strong coupling problems or instabilities, and in this way prevents HL Gravity from fully reproducing GR in the IR limit.

In Ref.[41] a new Covariant HL Gravity is formulated by Horava and Melby-Thompson (HM), which includes two additional nondynamical fields A and ν , together with a new $U(1)$ symmetry. In this model the $U(1)$ symmetry eliminates the extra scalar mode curing the strong coupling problems in the IR limit. Note, that in what follows we adopt the HM action of Ref. [41], in which the parameter λ is set equal to one ($\lambda = 1$). However, the $U(1)$ symmetry can not force the value of the parameter λ to be equal to 1, as an action, with the extended gauge symmetry and $\lambda \neq 1$, has been formulated in [42]. Also, HM theory reproduces many features of GR at long distances as it is shown initially in Ref.[41].

HL Gravity can be separated into two versions which are known as projectable and non-projectable. In the projectable version the lapse function N (see Eq. (2) below) depends only on the time coordinate, while in the

non-projectable version N is a function of both space and time coordinates. The Covariant HL Gravity considers the projectable case.

In this paper we study the most general case of spherically symmetric vacuum solutions in the framework of Covariant HL Gravity, for an action ¹ which includes all possible terms allowed by renormalizability requirement. We find that spherically symmetric solutions can be separated into two main classes: *(i)* solutions with nonzero radial shift function, and *(ii)* solutions with zero radial shift function.

We would like to note that Covariant HL Gravity, as it is formulated by Horava and Melby-Tomson (HM) in [41], incorporates an additional assumption for the field A . In particular, the field A is assumed as a part of an effective general relativistic metric in the IR limit, via the replacement $N \rightarrow N - A/c^2$. Spherically symmetric solutions, in the case *(ii)*, are consistent with observations if we adopt HM approach for A . On the other hand, in the case *(i)*, we see that A should be independent of the lapse function in order to achieve consistency with observation. We would like to warn the reader, that these two alternative but different views for the field A are examined both in this paper, in the framework of spherically symmetric vacuum solutions of cases *(i)* and *(ii)*.

The paper is organized as follows. In Sec.2 we summarize features of the Covariant HL Gravity. We consider then in Sec.3 the most general ansatz for spherically symmetric solutions, including a nonzero radial shift function, and we derive the equations of motion and the corresponding constraints. In Sec.4 we present the solutions for the two situations *(i)* and *(ii)*, and finally Sec.5 contains our conclusions. In Appendix A are presented some details on the geometrical interpretation of the auxiliary field A . In Appendix B, we present the minimal substitution approach of Ref. [42] which describes how matter couple with the auxiliary fields A and ν .

¹For the construction of the HL action, the so called "detailed balance principle" has been proposed [1]. The main advantage of this approach is the restriction of the large number of arbitrary couplings that appear in the action of the model. However, a more general way for constructing the action would be to include all possible operators which are compatible with the renormalizability [2, 43]; this implies that all operators with dimension less or equal to six are allowed.

2 Covariant Horava-Lifshitz Gravity

In this section we introduce the notation for the Covariant HL Gravity in the case of three spatial dimensions ($d = 3$), and we discuss the additional $U(1)$ symmetry of the model, as well as the role of the auxiliary nondynamical fields A and ν which are included in the action.

2.1 The action

This model, as the original HL Gravity, is characterized by an anisotropy between space and time dimensions

$$[t] = -z, \quad [x] = -1, \quad (1)$$

where z is an integer dynamical exponent. The action of the model is structured by a set of five fields: $N(t)$, $N_i(x, t)$, $g_{ij}(x, t)$, $A(x, t)$ and $\nu(x, t)$ ($i = 1, 2, 3$). Note that $N(t)$, $N_i(x, t)$, $g_{ij}(x, t)$ are the standard fields that appear in the Arnowitt, Deser and Misner (ADM) form of the space-time metric

$$ds^2 = -c^2 N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (2)$$

where c is the velocity of light, with dimension $[c] = z - 1$, and spatial components dx^i/dt ($i = 1, 2, 3$). In addition, N and N_i are the "lapse" and "shift" functions which are used in general relativity in order to split the space-time dimensions, and g_{ij} is the spatial metric of signature $(+, +, +)$. Note that here we are interested for the projectable version of the model which implies that the lapse function $N(t)$ depends only on the time parameter. For the dimensions of "lapse" and "shift" functions we obtain

$$[N] = 0, \quad [N_i] = z - 1. \quad (3)$$

The auxiliary fields $A(x, t)$ (potential) and $\nu(x, t)$ (prepotential²) are nondynamical fields which have to satisfy constraint equations. As we will see subsequently the existence of these fields is necessary in order to achieve invariance of the action under the extended Gauge symmetry: $U(1) \times Diff(\mathcal{M}, \mathcal{F})$. In addition, the dimensions of these fields are

$$[A] = 2z - 2, \quad [\nu] = z - 2. \quad (4)$$

²The terminology "potential" for $A(x, t)$ and "prepotential" for $\nu(x, t)$ has been introduced in the original work of [41]

The full action of the model is formulated as

$$S = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} \left\{ N [K_{ij} K^{ij} - K^2 - V + \nu \Theta^{ij} (2K_{ij} + \nabla_i \nabla_j \nu)] - A(R - 2\Omega) \right\}, \quad (5)$$

in which d is the spatial dimension ($D = d + 1 = 4$), κ^2 is an overall coupling constant with dimension $[\kappa^2] = z - d$, and the extrinsic curvature is

$$K_{ij} = \frac{1}{2N} \{ \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \}, \quad i, j = 1, 2, 3 \quad (6)$$

where the symbol Θ^{ij} is defined as

$$\Theta^{ij} = R^{ij} - \frac{1}{2} R g^{ij} + \Omega g^{ij}. \quad (7)$$

Note that this choice for $z = 3$ is an immediate consequence of power counting renormalizability request. In particular, the coupling κ^2 in the above action has dimension $[\kappa^2] = z - 3$, hence, if $z = 3$ the HL Gravity model is renormalizable, for $z > 3$ is super-renormalizable and for $0 < z < 3$ is non-renormalizable.

For the construction of the potential term V we will not follow the standard detailed balance principle, but we will use the more general approach [2, 43], according to which the potential term is constructed by including all possible renormalizable operators (relevant and marginal)³, that have dimension smaller than or equal to six, hence we write

$$V = V_{IR} + V_{R^2} + V_{R^3} + V_{\Delta R^2} \quad (8)$$

where

$$\begin{aligned} V_{IR} &= -c^2(R - 2\Lambda) \\ V_{R^2} &= -\alpha_1 R^2 - \alpha_2 R^{ij} R_{ij} \\ V_{R^3} &= -\beta_1 R^3 - \beta_2 R R^{ij} R_{ij} - \beta_3 R_i^j R_j^k R_k^i \\ V_{\Delta R^2} &= -\beta_4 R \nabla^2 R - \beta_5 \nabla_i R_{jk} \nabla^i R^{jk} \end{aligned} \quad (9)$$

The dimensions of the various terms in the Lagrangian are

$$[R] = 2, \quad [R^2] = 4, \quad [R^3] = [\Delta R^2] = 6, \quad (10)$$

³We have ignored terms which violate parity, see also [43].

where the symbol Δ is defined as $\Delta = g_{ij}\nabla^i\nabla^j$ ($i = 1, 2, 3$). In addition, we have used the notation R , R_{ij} and R_{ijkl} for the Ricci scalar, the Ricci and the Riemann tensors ($i, j = 1, 2, 3$), which correspond to the spatial 3D metric g_{ij} . Note that the term $R^{ijkl}R_{ijkl}$ does not appear in V_{R^2} , as the Weyl tensor in three dimensions automatically vanishes. In addition, c^2 , the couplings α_i ($i = 1, 2$), and β_j ($j = 1, \dots, 5$), have dimensions

$$[c^2] = 4 \quad , \quad [\alpha_i] = 2 \quad , \quad [\beta_j] = 0. \quad (11)$$

Finally, we would like to note that the potential terms, of Eqs. (8) and (9) above, has been considered previously for covariant HL gravity, in the case of cosmology, by the authors of Ref. [44].

2.2 The extended $U(1) \times Diff(\mathcal{M}, \mathcal{F})$ Gauge symmetry

The main motivation for considering models with an anisotropy between space and time dimensions type is the construction of a power-counting renormalizable Gravity model. However, in order to achieve normalizability, and simultaneously keep the time derivatives up to second order, we have to sacrifice the standard 4D diffeomorphism invariance of General Relativity, which is now restricted to

$$\delta t = f(t), \quad \delta x^i = \xi(t, x^j) \quad (12)$$

which is a foliation preserving diffeomorphism, $Diff(\mathcal{M}, \mathcal{F})$, where \mathcal{M} is the spacetime manifold, provided with a preferred foliation structure \mathcal{F} . In particular the fields N , N_i , g_{ij} transforms as:

$$\begin{aligned} \delta g_{ij} &= \partial_i \xi^k g_{jk} + \partial_j \xi^k g_{ik} + \xi^k \partial_k g_{ij} + f \dot{g}_{ij} \\ \delta N_i &= \partial_i \xi^j N_j + \xi^j \partial_j N_i + \xi^j g_{ij} + \dot{f} N_i + f \dot{N}_i \\ \delta N &= \xi^j \partial_j N + \dot{f} N + f \dot{N} \end{aligned} \quad (13)$$

However, the action of Eq. (5) has an additional symmetry, in particular it remains invariant under a $U(1)$ Gauge symmetry, according to which the fields of the model transform as

$$\begin{aligned} \delta_\alpha N &= 0 \\ \delta_\alpha g_{ij} &= 0 \\ \delta_\alpha N_i(x, t) &= N \nabla_i \alpha \\ \delta_\alpha A(x, t) &= \dot{\alpha} - N^i \nabla_i \alpha \\ \delta_\alpha \nu &= \alpha \end{aligned} \quad (14)$$

where α is an arbitrary spacetime function. Accordingly, the full symmetry of the action of Eq. (5) is the extended Gauge symmetry: $U(1) \times Diff(\mathcal{M}, \mathcal{F})$.

The extrinsic curvature term in the action of Eq. (5), if we assume only $Diff(\mathcal{M}, \mathcal{F})$ symmetry, could include an additional running coupling constant λ appearing as:

$$\mathcal{L}_K = K^{ij}K_{ij} - \lambda K^2, \quad (15)$$

In order to achieve agreement with General Relativity, we expect that RG flow in the IR leads the coupling λ to unity, but the exact mechanism for this remains unknown. There was a hope, in Ref. [41], that the Covariant model requires $\lambda = 1$ due to the $U(1)$ symmetry, hence the above mentioned problem, for the flow of λ in the IR, does not exist. On the other hand, in Ref. [42], it was shown that the action of Eq. (5) can be written in an invariant form under $U(1)$ symmetry, for an arbitrary value of λ , which implies that $U(1)$ symmetry can not fix the coupling λ (see Appendix B in the present paper).

Note that according to HM approach an infinitesimal $U(1)$ transformation on the fields A, ν, N, N^i , in the nonrelativistic limit, is equivalent to an infinitesimal diffeomorphism involving the time coordinate. As a consequence, the symmetry $U(1) \times Diff(\mathcal{M}, \mathcal{F})$ can be seen approximately as a $Diff(M, 4)$ symmetry of Standard General Relativity, see Ref. [41] and the discussion in our Appendix A.

2.3 Infrared limit

Although the additional $U(1)$ symmetry eliminates the extra degree of freedom, the IR limit of the HM theory, obtained after neglecting higher order in spatial curvature terms, does not coincide with General Relativity. Indeed, the action is then

$$S = \frac{1}{16\pi G} \int dx^0 d^3x \sqrt{g} \left\{ N [K_{ij}K^{ij} - K^2 + R + \nu \Theta^{ij} (2K_{ij} + \nabla_i \nabla_j \nu)] - A(R - 2\Omega) \right\} \quad (16)$$

where the time-like coordinate x_0 is defined as $x_0 = ct$, and the fields (in the above action) are rescaled according to Sec.5 in Ref. [41]. However, as discussed in [41], Covariant HL Gravity reproduces many features of general relativity for long distances.

3 Spherically symmetric solutions in Covariant HL Gravity

3.1 The metric

The starting point is the action [41], describing Gravity with anisotropic scaling:

$$S = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} \left\{ N [K_{ij} K^{ij} - K^2 - V + \nu \Theta^{ij} (2K_{ij} + \nabla_i \nabla_j \nu)] - A(R - 2\Omega) \right\}, \quad (17)$$

We consider the most general static spherically symmetric metric, of the form:

$$ds^2 = -c^2 N^2 dt^2 + \frac{1}{f(r)} (dr + n(r) dt)^2 + r^2 (\sin^2 \theta d\theta^2 + d\phi^2), \quad (18)$$

where $n(r) = N^r(r)$ is the radial component of shift functions, and $N_r = n(r)/f(r)$ since $g_{rr} = 1/f(r)$.

3.2 Constraints

The auxiliary fields ν and A lead to the following constraints

- The variation of S with respect to A gives $R - 2\Omega = 0$, or equivalently

$$R = -\frac{2}{r^2} (r f' + f - 1) = 2\Omega, \quad (19)$$

which imposes that the function $f(r)$ in the metric (18) is

$$f(r) = 1 - \frac{\Omega}{3} r^2 - \frac{2B}{r}, \quad (20)$$

where B is a constant of integration. As we will see, in the following sections, this constant of integration B is not interpreted necessary as the mass of the spherically compact object. Moreover, in what follows, we will set $\Omega = 0$, as we are looking for asymptotically flat solutions.

- The variation of S with respect to ν gives

$$\Theta^{ij}\nabla_i\nabla_j\nu + \Theta^{ij}K_{ij} = 0, \quad (21)$$

In what follows we will assume the Gauge fixing $\nu = 0$, then the above constraint gives $\Theta^{ij}K_{ij} = 0$, which is satisfied for spherically symmetric solutions as we will see in the following sections.

3.3 Equations of motion

We therefore start with the action

$$S \sim \int dt d^3x \sqrt{g} \left\{ N(T - V) - AR \right\}, \quad T = K_{ij}K^{ij} - K^2, \quad (22)$$

in which we have considered the Gauge fixing $\nu = 0$, and we have set $\Omega = 0$. The Lagrangian which corresponds to the action (22), has the form

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_T + \mathcal{L}_V - \frac{r^2}{\sqrt{f(r)}} AR, \\ \mathcal{L}_T &= \frac{N(t)r^2}{\sqrt{f(r)}} T(r, n, n', f, f'), \\ \mathcal{L}_V &= -\frac{N(t)r^2}{\sqrt{f(r)}} V(r, f, f', f'', f'''), \end{aligned} \quad (23)$$

where

$$\begin{aligned} T &= -\frac{8}{N^2(t)r^2}(n^2 f^2 + 2rn f^2 n' + rn^2 f f') \\ -\frac{r^2}{\sqrt{f(r)}} AR &= \frac{2}{\sqrt{f(r)}} A(r)(r f' + f - 1). \end{aligned} \quad (24)$$

The potential term of the Lagrangian, for zero cosmological term ($\Lambda = 0$) (as we are interested for solutions which are asymptotically flat), is taken as

$$\begin{aligned} V = & -c^2 R - \alpha_1 R^2 - \alpha_2 R^{ij} R_{ij} - \beta_1 R^3 - \beta_2 R R^{ij} R_{ij} \\ & - \beta_3 R^i{}_j R^j{}_k R^k{}_i - \beta_4 R \nabla^2 R - \beta_5 \nabla_i R_{jk} \nabla^i R^{jk}, \end{aligned} \quad (25)$$

in which

$$\begin{aligned}
R &= -\frac{2}{r^2}(rf' - 1 + f) \\
R^{ij}R_{ij} &= \frac{1}{2r^4}[3(rf')^2 + 4rf'(f - 1) + 4(f - 1)^2] \\
R^i{}_j R^j{}_k R^k{}_i &= -\frac{1}{4r^6}[5(rf')^3 + 6(rf')^2(f - 1) + 12rf'(f - 1)^2 + 8(f - 1)^3] \\
R\nabla^2 R &= \frac{2(rf' - 1 + f)}{r^6} \times \\
&\quad [2r^3 f''' f + r^3 f'' f' + 2r^2 f'' f + 2(1 - 3f)rf' - 4f(1 - f)] \\
\nabla_i R_{jk} \nabla^i R^{jk} &= \frac{f}{2r^6}[3(r^2 f'')^2 - 2r^3 f' f'' + 8r^2 f''(1 - f) + \\
&\quad + 5(rf')^2 + 16(1 - f)rf' + 24(1 - f)^2].
\end{aligned}$$

The Euler equation for n gives

$$\frac{d}{dr} \left(\frac{\partial \mathcal{L}_T}{\partial n'} \right) = \frac{\partial \mathcal{L}_T}{\partial n}, \quad (26)$$

from which we obtain

$$f'(r)n(r) = 0, \quad (27)$$

such that necessarily either $n(r) = 0$ or $f(r)$ is constant. The Euler equation for f is

$$\sum_{n=0}^3 (-1)^n \frac{d^n}{dr^n} \left(\frac{\partial \mathcal{L}}{\partial f^{(n)}} \right) = 0, \quad (28)$$

where $f^{(n)} = d^n f / dr^n$, hence we obtain

$$A' + \frac{A}{2r} \left(1 - \frac{1}{f} \right) + 4 \frac{fn(\sqrt{r}n)'}{N\sqrt{r}} = \mathcal{O}V, \quad (29)$$

where a prime denotes a derivative with respect to r and the differential operator \mathcal{O} is

$$\mathcal{O} = \frac{rN}{4f} - \frac{\sqrt{f}N}{2r} \sum_{n=0}^3 (-1)^n \frac{d^n}{dr^n} \left(\frac{r^2}{\sqrt{f}} \frac{\partial}{\partial f^{(n)}} \right). \quad (30)$$

After same algebra, for $f(r) = 1 - \frac{2B}{r}$ and $N = 1$, we obtain

$$\begin{aligned}
\mathcal{O}R &= \frac{B}{r(r-2B)} \\
\mathcal{O}R^2 &= 0 \\
\mathcal{O}R^{ij}R_{ij} &= \frac{B^2}{2r^4(r-2B)} \\
\mathcal{O}R^3 &= 0 \\
\mathcal{O}RR^{ij}R_{ij} &= -\frac{6B^2(11B-6r)}{r^7(r-2B)} \\
\mathcal{O}R^i{}_j R^j{}_k R^k{}_i &= -\frac{3B^2(50B-27r)}{2r^7(r-2B)} \\
\mathcal{O}R\nabla^2 R &= 0 \\
\mathcal{O}\nabla_i R_{jk}\nabla^i R^{jk} &= \frac{3B^2(40B-21r)}{2r^7(r-2B)}.
\end{aligned} \tag{31}$$

Finally, the variation of the action with respect to $N(t)$ gives the so called Hamiltonian constraint

$$\int_0^\infty dr \frac{r^2}{\sqrt{f(r)}} (T + V) = 0, \tag{32}$$

and, using a time redefinition, the lapse function $N(t)$ is set equal to unity ($N(t) = 1$).

4 Analytic solutions

We have to satisfy the following two equations of motion

$$f'(r)n(r) = 0 \tag{33}$$

$$A' + \frac{A}{2r} \left(1 - \frac{1}{f}\right) + \frac{4fn(\sqrt{r}n)'}{\sqrt{r}} = \mathcal{O}V, \tag{34}$$

and the Hamiltonian constraint

$$\int_0^\infty dr \frac{r^2}{\sqrt{f(r)}} (T + V) = 0, \tag{35}$$

while the constraint of the spatial curvature ($R = 0$) gives

$$f(r) = 1 - \frac{2B}{r}. \quad (36)$$

The constraint for ν (Eq. (21)) is verified for $\nu = 0$, what we will assume in what follows.

4.1 Nonzero radial shift function: $n(r) \neq 0$

If the shift function does not vanish, we see from the constraint (33) that f must be constant, and thus $B = 0$, such that

$$f = 1. \quad (37)$$

As a consequence, $V = 0$ (see Eqs. (31)), and Eq. (34) can be written as

$$rA' + 2(rn^2)' = 0, \quad (38)$$

while Eq. (35) gives the Hamiltonian constraint:

$$\int_0^\infty dr (rn^2)' = 0. \quad (39)$$

From Eqs. (38) and (39), we obtain then

$$\int_0^\infty dr rA'(r) = 0. \quad (40)$$

The above integral is convergent only if $A(r)$ has the following large and short distance asymptotic behavior

- $A(r) \simeq C_\infty + C_A r^{-b}$ for $r \rightarrow +\infty$, with $b > 1$;
- $A(r) \simeq C_0 + \bar{C}_A r^{-a}$ for $r \rightarrow 0$, with $a < 1$;

in which C_∞ , C_A , C_0 and \bar{C}_A are arbitrary constants. An integration by parts, of Eq. (40), leads to

$$\begin{aligned} \int_0^\infty dr rA'(r) &= \int_0^\infty dr r(A(r) - C_\infty)' \\ &= [r(A(r) - C_\infty)]_0^\infty - \int_0^\infty dr (A(r) - C_\infty) \\ &= - \int_0^\infty dr (A(r) - C_\infty) \end{aligned} \quad (41)$$

Note, that the fields A and $A + \text{const}$ are equivalent in the sense that they give the same n^2 from Eq. (38), hence without loss of generality we can set

$$C_\infty = \lim_{r \rightarrow \infty} A(r) = 0 \quad (42)$$

In this case the Hamiltonian constraint of Eq. (40) is equivalent to

$$\int_0^\infty dr A(r) = 0 \quad (43)$$

In what follows we consider separately two cases 1) $A(r) = 0$ which is the minimal choice, 2) $A(r)$ is a function which satisfy the Hamiltonian constraint of Eq. (43) (or the equivalent equation (40)).

4.1.1 First case: $A=0$

When $A = 0$, the constraint of Eq. (43) is satisfied, hence from Eq. (38) we obtain

$$n(r) = \pm \sqrt{\frac{C_M}{r}}, \quad C_M = 2GMc^2 \quad (44)$$

where C_M is a constant of integration. Thus the metric of Eq. (18) can be written as

$$ds^2 = -c^2 dt^2 + \left(dr \pm \sqrt{\frac{2GMc^2}{r}} dt \right)^2 + r^2 (\sin^2 \theta d\theta^2 + d\phi^2), \quad (45)$$

which is the Schwarzschild solution in Painlevé-Gullstrand coordinates, see for example Ref. [4] and references therein. Note that in this case, the Newtonian potential $\phi(r)$ ($g_{00} = 1 + 2\phi(r)$) is proportional to the square of the radial shift function, according to the equation

$$\phi(r) = -\frac{n^2(r)}{2c^2} = -\frac{GM}{r} \quad (46)$$

However, we observe that the above expression for the Newtonian potential is not $U(1)$ invariant. In particular it is the Newton Law for particular Gauge choice $\nu = 0$. We can correct this situation if we take into account the coupling between shift functions N_i and matter. In Ref. [42] (see also

Appendix B) we see that N_i couple with matter in the $U(1)$ invariant form $N_i - N\nabla_i\nu$, such that the Newtonian potential should be modified as

$$\phi(r) = -\frac{(n(r) - \nabla_r\nu(r))^2}{2c^2} = -\frac{GM}{r}. \quad (47)$$

The above expression is $U(1)$ invariant, and for the Gauge choice $\nu = 0$ we recover Eq. (46).

Finally, in the case of the initial HL Gravity model with projectability condition, spherically symmetric solutions have been studied in Ref. [4]. The solution we present in this section, for Covariant HL Gravity, is also a solution of HL Gravity without the $U(1)$ symmetry, but these two models have not the same full spectrum of solutions. Also, spherically symmetric solutions with nonzero energy momentum tensor (stars), in the case of projectability condition, have been studied in Ref. [15], and spherically symmetric star solutions are studied in Ref. [17]: the main conclusion is that a spherically-symmetric star should include a time-dependent region near the center.

4.1.2 Second case: $\int_0^\infty dr A(r) = 0$

In this situation, the most general solution of Eq. (38) is

$$\begin{aligned} n^2(r) &= \frac{\tilde{C}_M}{r} - \frac{1}{2}A(r) + \frac{1}{2r} \int_0^r d\rho A(\rho) \\ &= \frac{\tilde{C}_M}{r} - \frac{1}{2}A(r) - \frac{1}{2r} \int_r^\infty d\rho A(\rho), \end{aligned} \quad (48)$$

where \tilde{C}_M is a constant, and we will consider separately the two cases: **(a)** $\tilde{C}_M = 0$ and **(b)** $\tilde{C}_M \neq 0$.

(a) We have for $\tilde{C}_M = 0$

$$n^2(r) = -\frac{1}{2}A(r) - \frac{1}{2r} \int_r^\infty d\rho A(\rho). \quad (49)$$

Since $A(r)$ behaves as $A(r) \simeq -C_A r^{-b}$ for large distances, Eq. (49) gives a modified Newtonian potential when $r \rightarrow \infty$

$$\phi(r) = -\frac{n^2(r)}{2c^2} \simeq -\frac{C_\phi}{r^b}, \quad b > 1, \quad (50)$$

in which

$$C_\phi = \frac{bC_A}{4c^2(b-1)}, \quad (51)$$

where $C_A > 0$ is dimensionful, in order for the potential to have the correct dimensionality, and C_ϕ is interpreted as the mass of the compact object. Also, because $n^2 \geq 0$, we have the additional constraint:

$$\int_0^r A(\rho) d\rho \geq rA(r), \quad (52)$$

and to show a function $A(r)$ satisfying the above constraints can be found, we give here two examples.

A first example of function $A(r)$, which satisfies the condition (41), is

$$A_1(r) = -\frac{C_A}{1+r^b} \left(1 - \frac{\gamma_1}{r^{1/b}}\right), \quad b > 1 \quad (53)$$

where

$$\gamma_1 = \frac{\sin\left(\frac{\pi}{b} - \frac{\pi}{b^2}\right)}{\sin\left(\frac{\pi}{b}\right)}. \quad (54)$$

The condition (52) is then satisfied for all r only if $C_A > 0$, such that the potential (50) is negative, as expected. Note that this function is singular for $r = 0$, and another example of function $A(r)$ which is regular at the origin and which satisfies the constraints, is:

$$A_2(r) = -\frac{C_A}{1+r^{b+1}} (r - \gamma_2), \quad b > 1 \quad (55)$$

where

$$\gamma_2 = \frac{\sin\left(\frac{\pi}{b+1}\right)}{\sin\left(\frac{2\pi}{b+1}\right)}. \quad (56)$$

Although we have not check it, by choosing b appropriately closely to unity $b \approx 1$ it may be possible the above two solutions to pass solar system tests. However, the purpose of this paper is not to set constraints on the parameter b , hence the topic of constructing solutions which can satisfy solar system tests is left for future investigations. Also, note that the leading order of large distance behavior for the radial shift function $n(r)$ (or $A(r)$) is fixed by the requirement of recovery of Newton Law in the large distance limit, but for small distances there is an ambiguity in the exact shape of $n(r)$ (or

$A(r)$), as we see from the above two examples.

(b) For $\tilde{C}_M \neq 0$ we have

$$n^2(r) = \frac{\tilde{C}_M}{r} - \frac{1}{2}A(r) - \frac{1}{2r} \int_r^\infty d\rho A(\rho), \quad (57)$$

which corresponds to a qualitatively different situation from the one where $\tilde{C}_M = 0$. Here the constant \tilde{C}_M is proportional to the mass of the spherical compact object $\tilde{C}_M = C_M = 2GMc^2$, and the auxiliary field A determines the subleading behavior in the asymptotic expansion of n^2 for large r . As in the previous case ($\tilde{C}_M = 0$), we can choose suitably the field $A(r)$ in order to satisfy the Hamiltonian constraint of Eq. (41), and the restriction $n^2 > 0$. It is explained in [16] that solar system tests requires a large distance asymptotic behavior for A of the form:

$$A(r) \simeq \frac{\tilde{C}_A}{r^b} + \dots, \quad \text{when } r \rightarrow \infty, \quad (58)$$

where the exponent satisfies $b \geq 3$, and the dots represent higher order powers of $1/r$. We find here that the following choice for $A(r)$ has the required asymptotic behavior, and satisfies the corresponding constraints (the Hamiltonian constraint of Eq. (41) and the requirement $n^2 \geq 0$),

$$A_3(r) = \frac{\tilde{C}_A}{1 + r^{b_1}} (1 - \gamma_3 r^{b_2}), \quad (59)$$

with $3 \leq b_1, \quad 3 \leq b_1 - b_2, \quad \text{and} \quad -1 < b_2 \neq 0$

where \tilde{C}_A is a dimensionful constant and

$$\gamma_3 = \frac{\sin\left(\frac{\pi}{b_1} + \frac{\pi b_2}{b_1}\right)}{\sin\left(\frac{\pi}{b_1}\right)}. \quad (60)$$

Hence solar system tests do not necessarily impose $A = 0$, and the choice for the function A remains an open question. Furthermore, according to the above examples we see that there is a freedom in the choice of A , which correspond to different spherically symmetric solutions, in contrast to standard General Relativity for which spherical symmetry (in the absent of matter) leads to Schwarzschild geometry.

4.2 Zero radial shift function: $n(r) = 0$

In this situation, $f(r) = 1 - 2B/r$, with $B \neq 0$, and the evolution equation (29) for A gives

$$\begin{aligned} \frac{dA}{dr} - \frac{BA}{r(r-2B)} = & -\frac{c^2 B}{r(r-2B)} - \frac{\alpha_2 B^2}{2r^4(r-2B)} \\ & - \frac{6\beta_2 B^2}{(r-2B)} \left(\frac{6}{r^6} - \frac{11B}{r^7} \right) \\ & - \frac{3\beta_3 B^2}{2(r-2B)} \left(\frac{27}{r^6} - \frac{50B}{r^7} \right) \\ & + \frac{3\beta_5 B^2}{2(r-2B)} \left(\frac{21}{r^6} - \frac{40B}{r^7} \right). \end{aligned} \quad (61)$$

The solution of this equation is

$$\begin{aligned} A(r) = & c^2 + A_0 \sqrt{1-2x} - \frac{\alpha_2}{10B^2} (-2 + 2x + x^2 + x^3) + \frac{6\beta_2}{B^4} x^6 \\ & + \frac{\beta_3}{11B^4} \left(-\frac{4}{7} + \frac{4}{7}x + \frac{2}{7}x^2 + \frac{2}{7}x^3 + \frac{5}{14}x^4 + \frac{x^5}{2} + 75x^6 \right) \\ & - \frac{3\beta_5}{11B^4} \left(-\frac{4}{7} + \frac{4}{7}x + \frac{2}{7}x^2 + \frac{2}{7}x^3 + \frac{5}{14}x^4 + \frac{x^5}{2} + 20x^6 \right), \end{aligned} \quad (62)$$

where $x = B/r$ and A_0 is a constant of integration. If we set $B = GM$ ⁴, the solution of Eq. (62) can be written as an expansion in $x = M/r$ (from now on it is convenient to set $G = 1$)

$$\begin{aligned} A(r) = & c^2 + A_0 + \frac{\alpha_2}{5M^2} - \frac{4\beta_3}{77M^4} + \frac{12\beta_5}{77M^4} \\ & + \left(x + \frac{x^2}{2} + \frac{x^3}{2} \right) \left(-A_0 - \frac{\alpha_2}{5M^2} + \frac{4\beta_3}{77M^4} - \frac{12\beta_5}{77M^4} \right) \\ & + x^4 \left(-\frac{5A_0}{8} + \frac{5\beta_3}{154M^4} - \frac{15\beta_5}{154M^4} \right) \\ & + x^5 \left(-\frac{7A_0}{8} + \frac{\beta_3}{22M^4} - \frac{3\beta_5}{22M^4} \right) \end{aligned} \quad (63)$$

⁴In the case of solution with zero shift function the constant of integration B is proportional to the mass of the compact object.

$$+x^6 \left(-\frac{21A_0}{8} + \frac{6\beta_2}{M^4} + \frac{75\beta_3}{11M^4} - \frac{60\beta_5}{11M^4} \right) + \mathcal{O}(x^7).$$

in which the constant of integration A_0 is chosen to vanish the constant term

$$c^2 + A_0 + \frac{\alpha_2}{5M^2} - \frac{4\beta_3}{11M^4} + \frac{12\beta_5}{77M^4} = 0, \quad (64)$$

Now $A(r)$ can be split to an IR and an UV part according to the equation:

$$A(r) = A_{IR}(r) + A_{UV}(r) \quad (65)$$

where we have set

$$\begin{aligned} A_{IR}(r) &= c^2 (1 - \sqrt{1 - 2x}) \\ A_{UV}(r) &= - \left(\frac{\alpha_2}{5M^2} - \frac{4\beta_3}{11M^4} + \frac{12\beta_5}{77M^4} \right) \sqrt{1 - 2x} \\ &\quad - \frac{\alpha_2}{10M^2} (-2 + 2x + x^2 + x^3) + \frac{6\beta_2}{M^4} x^6 \\ &\quad + \frac{\beta_3}{11M^4} \left(-\frac{4}{7} + \frac{4}{7}x + \frac{2}{7}x^2 + \frac{2}{7}x^3 + \frac{5}{14}x^4 + \frac{x^5}{2} + 75x^6 \right) \\ &\quad - \frac{3\beta_5}{11M^4} \left(-\frac{4}{7} + \frac{4}{7}x + \frac{2}{7}x^2 + \frac{2}{7}x^3 + \frac{5}{14}x^4 + \frac{x^5}{2} + 20x^6 \right) \end{aligned} \quad (66)$$

4.2.1 The potential interpretation of A

According to the original formulation of covariant HL gravity by Horava Melby (HM) in [41] the field A has a particular role in the IR limit: more specifically it is promoted as a part of an effective general relativistic metric via the replacement

$$N \rightarrow N - \frac{A_{IR}}{c^2}. \quad (68)$$

Hence the spacetime geometry is determined effectively by the following metric

$$ds^2 = -c^2 \left(N^2 - \frac{N^i N_i + 2A_{IR}N}{c^2} \right) dt^2 + 2N^i dx_i dt + g_{ij} dx^i dx^j, \quad (69)$$

which is realized in the nonrelativistic limit $c \rightarrow \infty$, by dropping higher order terms in $1/c^2$, as it is shown initially in [41] (for details see also Appendix A).

In the above effective metric of Eq. (69) we have included only the A_{IR} part of A which is consistent with HM approach. On other hand the A_{UV} part of A can not be included in this metric, as it is subdominant in the $1/c^2$ expansion. Note that an investigation of the physical role of A_{UV} (in HM approach) is beyond the scope in this paper.

It is worth noting that, in the case of solutions with zero shift function ($N_i = 0$ and $N = 1$), the HM approach allows a potential interpretation for the field A . If we take into account that in the IR limit ($r \rightarrow \infty$)

$$g_{00} = 1 + 2\phi + \dots$$

and compare with Eq. (69) above, we obtain that the Newtonian potential $\phi(r)$ is related with $A(r)$ according to the equation

$$\phi(r) = -\frac{A_{IR}(r)}{c^2} = -\frac{GM}{r} + O\left(\frac{GM}{r}\right)^2. \quad (70)$$

Also the effective metric of Eq. (69) in the IR limit, in the case of solutions with zero shift function, can be written as

$$ds_{eff}^2 = -c^2 \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (71)$$

It is clear that the above metric mimics Schwarzschild geometry in the IR, such that the metric of Eq. (71) passes solar system tests, and no restriction on Horava couplings is necessary (for details see also [41]). At this point, it should be emphasized that the geometric approach for A is an independent assumption of HM theory, however in this paper we use it mainly to make our solutions (with zero shift function) physically relevant. Finally, the full expression of $A(r)$ for $r \rightarrow \infty$, is given by the expression

$$\begin{aligned} \frac{A(r)}{c^2} &= \frac{M}{r} + \frac{M^2}{2r^2} + \frac{M^3}{2r^3} \\ &+ \left(\frac{5}{8} + \frac{\alpha_2}{8M^2c^2}\right) \left(\frac{M^4}{r^4} + \frac{7M^5}{5r^5}\right) \\ &+ \left(\frac{21}{8} + \frac{21\alpha_2}{40M^2c^2} + \frac{6\beta_2}{M^4c^2} + \frac{147\beta_3}{22M^4c^2} - \frac{111\beta_5}{22M^4c^2}\right) \frac{M^6}{r^6} \\ &+ \mathcal{O}(M^7/r^7). \end{aligned} \quad (72)$$

We note that the first two corrections, in M^2/r^2 and M^3/r^3 , are independent of the Horava-Lifshitz couplings. Also, one can see that corrections of high order derivatives vanish in the nonrelativistic limit ($c \rightarrow \infty$).

4.2.2 The Hamiltonian constraint

In the situation where $n = 0$, the Hamiltonian constraint reads

$$\int_0^\infty dr \frac{r^2}{\sqrt{f}} V = 0, \quad (73)$$

For the case we consider, $f(r) = 1 - 2M/r$, the potential V is

$$V = \frac{6\alpha_2 M^2}{r^6} - \frac{6\beta_3 M^3}{r^9} - \frac{90\beta_5 M^2}{r^9} (r - 2M), \quad (74)$$

In order to keep the integrand function in Eq. (73) real, we have to introduce a spherical gap in space, centered on the black hole and including the horizon, assuming that the radial coordinate r has the minimum value L , where $L \geq 2M$. An equivalent alternative⁵ is for example to introduce the new coordinate $u \geq 0$, defined by $r = \sqrt{u^2 + L^2}$, and to express the whole problem in terms of u instead of r . Note that space in this situation is still simply connected, such that this non-vanishing length L does not introduce topological defects. In this case we have

$$\begin{aligned} \int_L^\infty dr \frac{r^2}{\sqrt{f}} V &= 6M^2 \int_L^\infty \frac{dr}{\sqrt{r-2M}} \left(\frac{\alpha_2}{r^{7/2}} - \frac{\beta_3 M}{r^{13/2}} + \frac{15\beta_5}{r^{13/2}} (r - 2M) \right) \\ &= M^{-1} \alpha_2 C_2 - M^{-3} \beta_3 C_3 - M^{-3} \beta_5 C_5, \end{aligned} \quad (75)$$

where the dimensionless functions C_2, C_3, C_5 are given by the equations

$$\begin{aligned} C_2(y) &= \frac{4}{5} - \frac{2\sqrt{y-2}}{5y^{5/2}} (3 + 2y + 2y^2) \\ C_3(y) &= \frac{16}{231} - \frac{2\sqrt{y-2}}{231y^{11/2}} (63 + 35y + 20y^2 \\ &\quad + 12y^3 + 8y^4 + 8y^5) \\ C_5(y) &= \frac{16}{77} - \frac{2(y-2)^{3/2}}{77y^{11/2}} (315 + 140y + 60y^2 \\ &\quad + 24y^3 + 8y^4), \end{aligned} \quad (76)$$

⁵The authors would like to thank Alex Kehagias for this suggestion.

where we have set $y = L/M$. The Hamiltonian constraint leads to the following algebraic equation

$$C_2(y) - \tilde{\beta}_3 C_3(y) - \tilde{\beta}_5 C_5(y) = 0, \quad (77)$$

in which we have set $\tilde{\beta}_3 = \beta_3 \alpha_2^{-1} M^{-2}$ and $\tilde{\beta}_5 = 15 \beta_5 \alpha_2^{-1} M^{-2}$. We can determine the lower limit L by solving numerically Eq. (77), for certain values of $\tilde{\beta}_3$ and $\tilde{\beta}_5$. Note, that a detailed investigation of Eq. (77) is rather involved and unnecessary for this study. However, we have performed computations for specific values of $\tilde{\beta}_3$ and $\tilde{\beta}_5$, and we present our results in Fig. 1. In this figure he have plotted the function

$$H(y) = C_2(y) - \tilde{\beta}_3 C_3(y) - \tilde{\beta}_5 C_5(y)$$

versus y , for fixed $\tilde{\beta}_3 = 5$, and several values of $\tilde{\beta}_5 = -1, 0, 1, 1.6, 1.8, 2, 2.2, 2.5$. We observe that for $\tilde{\beta}_5 \leq 1.6$ Eq. (77) has no real solutions, for $1.6 < \tilde{\beta}_5 \leq 1.8$ Eq. (77) has two real solutions, and for $1.8 < \tilde{\beta}_5$ Eq. (77) has one real solution. This situation seems to be quite general, as we have performed computations for other values of $\tilde{\beta}_3$, which are not presented in this paper, and we observed the same behavior for the function $H(y)$. According to the above mentioned results, there is a region of the parameter space for which there is no lower limit L , which vanish the Hamiltonian constraint of Eq. (77). Also, there is a region in which there is an arbitrariness in the choice of L , as Eq. (77) has two distinct solutions for $y = L/M$. Finally, there is a region for which L is determined uniquely by the couplings $\tilde{\beta}_3$, and $\tilde{\beta}_5$ of the spherical symmetric object, as Eq. (77) has one real solution.

It would be nice if we could determine the range of r by physical and geometrical considerations. However, the spherically symmetric solutions with zero shift function should satisfy the Hamiltonian constraint. Note that the potential in Eq. (74) diverges for $r = 0$, so that it is unavoidable to introduce a lower limit L in the corresponding integral of Eq. (75). As we show explicitly above, this lower limit is determined by the free parameters of the model, if we try to satisfy the Hamiltonian constraint of Eq. (75) (or of Eq. (73)). It seems that the range of acceptable values for r can not be determined geometrically, for spherically symmetric solutions with zero shift function in Covariant Horava Lifshitz Gravity, unlike the case of standard black hole solutions in General Relativity. However, we think that this topic needs further investigation.

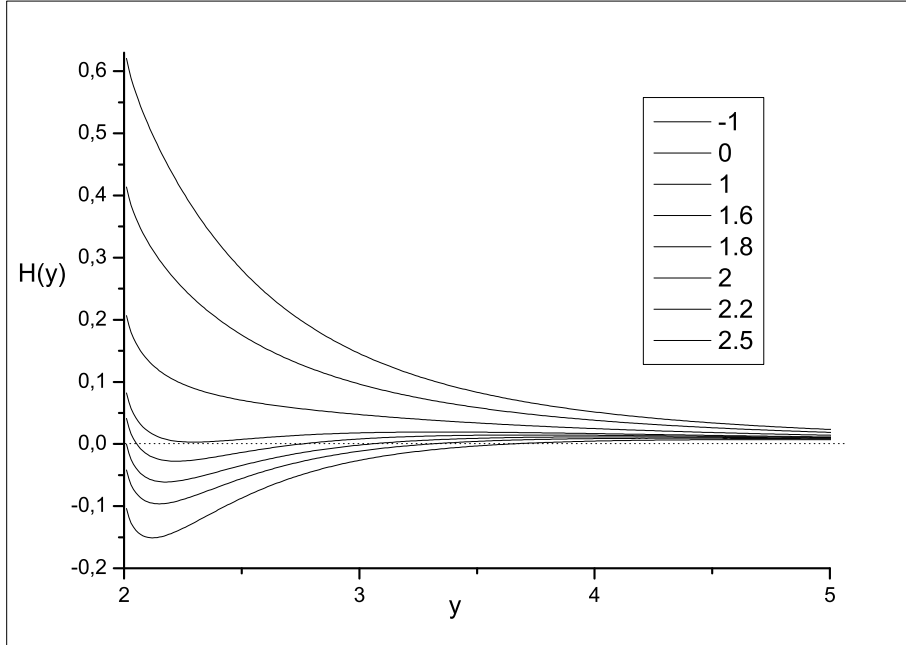


Figure 1: $H(y) = \alpha_2 C_2(y) - \tilde{\beta}_3 C_3(y) - \tilde{\beta}_5 C_5(y)$ versus y , for $\tilde{\beta}_3 = 5$ and increasing values for $\tilde{\beta}_5$, from -1 (upper curve), 0, 1, 1.6, 1.8, 2, 2.2, 2.5 (lower curve).

5 Conclusions

We studied the most general case of spherically symmetric vacuum solutions in the framework of Covariant Horava Lifshitz Gravity. More specifically, we found two classes of spherically symmetric solutions: *(i)* with nonzero radial shift function (see Sec. 4.1) and *(ii)* with zero radial shift function (see Sec. 4.2). We show that solutions of class *(ii)* becomes physically relevant if we adopt the position of HM for the geometrical role of A in the IR, while solutions of class *(i)* are consistent with observations only if A is independent of spacetime geometry.

In the case of solutions with nonzero radial shift function we would like to mention the freedom of the non-dynamical field $A(r)$, although the latter has to satisfy constraint equations, as we see in Sec. 4.1. In particular, $A(r)$ can be a constant, or it must satisfy a set of constraints which are presented

in Sec. 4.1.2. This class of solutions permits an alternative interpretation for the field A other than that of HM, as the Newtonian potential for large distances is recovered by the radial shift function. In the latter case - situation (ii) - one needs to interpret geometrically the $U(1)$ transformation as a space time symmetry, in order to recover the Newtonian potential in the non-relativistic limit, which in the situation (i) is not necessary.

Note that, in order to recover the standard Schwarzschild geometry, expressed in Painlevé-Gullstrand coordinates, one needs to take the solution for which A is constant. For the other solutions, where $A(r)$ is not constant and satisfies the constraints in Sec.4.1.2, we obtain solutions which obey a modified power law for the Newtonian potential, with a leading term $\phi(r) \sim r^{-b}$ and $b > 1$. These solutions may possibly be consistent with solar system tests if b is chosen appropriately closely to unity ($b \approx 1$), but this would still need to be checked. However, in Sec.4.1.2 we present a solution, which can satisfy solar system tests and corresponds to a nonconstant field A (see Eq. (59)), and a spacetime geometry which agree with Schwarzschild metric only asymptotically.

Note that, although the leading order of large distance behavior for the radial shift function of $A(r)$ is fixed by the requirement of recovery of Newton Law in the large distance limit, for small distances there is an ambiguity in the exact shape of $A(r)$, as we see from the examples in Sec. 4.1.2.

Finally, for solutions with non-zero radial component of shift functions, higher order spatial curvature corrections in the action do not play a role, and the coupling constants in Eq. (25) do not appear in this class of solutions.

In the case of zero radial shift function, in contrast with the previous case, higher order curvature terms do not vanish and, as a result, the corresponding higher order couplings appear in the solution for the auxiliary field A . In Sec. 4.2.1 we explain why only the IR part of A possesses a geometric interpretation in the IR, while the role of the UV part of A , in the framework of HM view, remains ambiguous.

In addition, this situation is characterized by a difficulty related to the Hamiltonian constraint. In order to satisfy the latter, we had to assume the existence of a spherical gap in space (with radius L), centered on the spherical compact object. The radius L of the gap can be determined by an algebraic equation which can be solved only numerically, as it is shown in Sec. 4.2.2. Note, that there are regions of the free parameter space for which this equation has no solutions, or has more than one solution (in particular two). In this sense, for the case of two solutions, there is an arbitrariness in the choice

of L . In addition, in this class the auxiliary field A is determined unambiguously by the corresponding constraint (for details see Secs. 3 and 4.2). Also, note that this class of solutions can pass solar system tests, failing to impose restrictions to Horava couplings.

The main conclusion of this paper, is that beyond the geometrical approach of HM, there is an alternative view for the field A which is also consistent with observations. In the latter case the field A is considered to be independent of the spacetime geometry, in contrast to HM theory, and the Newton law is reproduced by the nonzero shift function of the solutions of class (i). Finally, note that the field A is not completely fixed for solutions of class (i), so a study on this topic may be a topic for future investigation.

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Note added in proof We would like to acknowledge that the alternative view for the field A has been also proposed independently in the parallel paper [16], which appeared on the arXiv almost simultaneously with this work.

Appendix A: Geometrical interpretation of the $U(1)$ symmetry

We consider the standard metric in the ADM form

$$ds^2 = -c^2 \left(N(t)^2 - \frac{N^i N_i}{c^2} \right) dt^2 + 2N^i dx_i dt + g_{ij} dx^i dx^j, \quad (78)$$

in which the lapse function $N(t)$ is assumed to be only a function of time. We can promote the auxiliary field $A(x, t)$ as a part of spacetime geometry, and simultaneously introduce a new spacetime dependent the lapse function, by performing the replacement

$$N(t) \rightarrow N(t) - \frac{A(x, t)}{c^2}.$$

The above metric can be written as

$$ds^2 = -c^2 \left(N^2 - \frac{N^i N_i + 2AN}{c^2} \right) dt^2 + 2N^i dx_i dt + g_{ij} dx^i dx^j, \quad (79)$$

where we have dropped the higher order term A^2/c^4 , as we are interested for the nonrelativistic limit $c \rightarrow \infty$.

It is easy to see, in the limit where $c \rightarrow \infty$, that the metric (79) is invariant under a spacetime dependent reparametrization of time, according to

$$t' = t + \frac{\varepsilon(x, t)}{c^2}, \quad x' = x, \quad (80)$$

if this transformation is accompanied by the $U(1)$ symmetry

$$\begin{aligned} N'_i(x, t) &= N_i(x, t) + N^2 \nabla_i \varepsilon \\ A'(x, t) &= A(x, t) + \dot{\varepsilon} N + \varepsilon \dot{N} - N N^i \nabla_i \varepsilon, \end{aligned} \quad (81)$$

where

$$\varepsilon(x, t) = \frac{\alpha(x, t)}{N}.$$

(Note that, under this $U(1)$ transformation, the fields $N(t)$ and $g_{ij}(x, t)$ remain unaltered). Indeed, if we consider the general relativistic metric in the t' coordinate

$$ds^2 = -c^2 \left(N^2 - \frac{N^i N'_i + 2A'N}{c^2} \right) (dt')^2 + 2N^i dx_i dt' + g_{ij} dx^i dx^j, \quad (82)$$

where all the fields depend on t' , the $U(1)$ transformation of Eqs. (81), together with

$$N(t') = N \left(t + \frac{\varepsilon(x, t)}{c^2} \right) \simeq N(t) + \frac{\dot{N} \varepsilon(x, t)}{c^2}, \quad (83)$$

$$dt' = \left(1 + \frac{\dot{\varepsilon}}{c^2} \right) dt + \frac{\nabla_i \varepsilon dx^i}{c^2} + O \left(\frac{1}{c^4} \right), \quad (84)$$

show that the metrics (82) and (79) are equivalent, up to higher order terms in $1/c$.

Appendix B: $U(1)$ symmetry by a minimal substitution

In Ref. [42] is proposed a minimal substitution mechanism, which can be used to extend the Gauge symmetry of any $Diff(\mathcal{M}, \mathcal{F})$ invariant action.

This mechanism is based on the observation that the following quantities are invariant under the $U(1)$ transformation,

$$\delta_a(N_i - N\nabla_i\nu) = 0, \quad \delta_a(A - a) = 0 \quad (85)$$

where a is defined as

$$a = \dot{\nu} - N^j \nabla_j \nu + \frac{N}{2} \nabla_j \nu \nabla^j \nu. \quad (86)$$

We consider, a $Diff(\mathcal{M}, \mathcal{F})$ invariant action of the form:

$$S[N, N_i, g_{ij}, \psi_n] = S_{HL}[N, N_i, g_{ij}] + S_m[N, N_i, g_{ij}, \psi_n], \quad (87)$$

in which the first term S_{HL} is the standard HL action

$$S_{HL}[N, N_i, g_{ij}] = \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} N [K_{ij} K^{ij} - \lambda K^2 - V], \quad (88)$$

and the second term S_m represents the interaction between the external fields ψ_n and the gravitational fields N, N_i, g_{ij} . The fields ψ_n may be, for example, scalar or vector fields. The action $S[N, N_i, g_{ij}, \psi_n]$ can be promoted to a manifestly $U(1)$ invariant action $\hat{S}[N, N_i, g_{ij}, \psi_n, A, \nu]$, if we perform the replacement $N_i \rightarrow N_i - N\nabla_i\nu$, and simultaneously add an extra term which depends only on $(A - a)$, according to the equation:

$$\hat{S}[N, N_i, g_{ij}, \psi_n, A, \nu] = S[N, N_i - N\nabla_i\nu, g_{ij}, \psi_n] + \int dt d^3x \sqrt{g} Z(\psi_n, g_{ij})(A - a) \quad (89)$$

where $Z(\psi_n, g_{ij})$, with dimension $[Z] = 2$, is the most general operator which is invariant under $Diff(\mathcal{M}, \mathcal{F})$ and respects renormalizability requirements. This minimal substitution mechanism gives naturally the answer of how one can couple matter with the auxiliary fields A and ν .

As shown in Ref. [42], in the absence of the external field ψ_n , if we set $\lambda = 1$, the manifestly $U(1)$ invariant action $\hat{S}[N, N_i, g_{ij}, 0, A, \nu]$ is identical with the action of Covariant HL Gravity, as it is given by Eq. (5) above, or equivalently we can write:

$$\begin{aligned} \hat{S} &= S_{HL}[N, N_i - N\nabla_i\nu, g_{ij}] - \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} (A - a)(R - \Omega) \\ &= \frac{2}{\kappa^2} \int dt d^3x \sqrt{g} \left\{ N [K_{ij} K^{ij} - K^2 - V + \nu \Theta^{ij} (2K_{ij} + \nabla_i \nabla_j \nu)] \right. \\ &\quad \left. - A(R - 2\Omega) \right\}, \end{aligned} \quad (90)$$

Note that, if we set $\psi_n = 0$, we obtain

$$Z(0, g_{ij}) = -\frac{2}{\kappa^2}(R - \Omega).$$

Finally, we observe that this derivation of the Covariant HL action does not require $\lambda = 1$, which implies that $U(1)$ symmetry can not force the value of λ to be $\lambda = 1$. However, for $\lambda \neq 1$, the action of Eq. (90) above is modified by a term of the form:

$$S_\lambda = \frac{2}{\kappa^2} \int dt d^d x \sqrt{g} N (1 - \lambda) (K + \triangle \nu)^2 . \quad (91)$$

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